Evolution equation for a model of surface relaxation in complex networks

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In this paper we derive analytically the evolution equation of the interface for a model of surface growth with relaxation to the minimum (SRM) in complex networks. We were inspired by the disagreement between the scaling results of the steady state of the fluctuations between the discrete SRM model and the Edward-Wilkinson process found in scale-free networks with degree distribution $P(k) \sim k^{-\lambda}$ for $\lambda < 3$ [Pastore y Piontti *et al.*, Phys. Rev. E **76**, 046117 (2007)]. Even though for Euclidean lattices the evolution equation is linear, we find that in complex heterogeneous networks nonlinear terms appear due to the heterogeneity and the lack of symmetry of the network; they produce a logarithmic divergency of the saturation roughness with the system size as found by Pastore y Piontti *et al.* for $\lambda < 3$.

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During the last few years the study of complex networks has moved its focus from the study of their topology to the dynamic processes occurring on the underlying network. This is because many physical and dynamic processes use complex networks as substrates. Recently, many studies of dynamic processes on networks, such as epidemic spreading [2], traffic flow [3,4], cascading failure [5], and synchronization [6,7], have demonstrated the importance of the topology of the substrate network in the dynamic process. There exists much evidence that many real networks possess a scale-free (SF) degree distribution characterized by a power law tail given by $P(k) \sim k^{-\lambda}$, where $k_{max} \geq k \geq k_{min}$ is the degree of a node, k_{max} is the maximum degree, k_{min} is the minimum degree, and λ measures the broadness of the distribution [8]. Almost all the studies on networks regarded the links or nodes as identical. However, in real networks the links or nodes are not identical but have some "weight." As examples, the links between computers in the internet network have different capacities or bandwidths, resistor networks can have different values of resistance [4], and the airline network links connecting pairs of cities in direct flights have different numbers of passengers. Many theoretical studies have been carried out on weighted networks [4,9]. Recently, several studies on real networks with weights on the links, such as the world-wide airport networks and the *Escherichia coli* metabolic networks [10], have shown that the weights are correlated with the network topology and this dramatically changes the transport through them [7,11]. For instance, in synchronization problems, which are very important in brain networks [12], networks of coupled populations in the synchronization of epidemic outbreaks [13], and the dynamics and fluctuations of task completion landscapes in causally constrained queuing networks [14], the weights could have dramatic consequences for the synchronization [7]. Synchronization problems deal with optimization of the fluctuations of some scalar field h. The system will be optimally synchronized when the fluctuations are minimized. The general treatment to analyze the fluctuations of these processes is to map them into a problem of nonequilibrium surface growth via an Edwards-Wilkinson (EW) process on the corresponding network [15]. Given a scalar field h on the nodes that represents the interface height at each node, the fluctuations are characterized by the average roughness $W(t)$ of the interface at time t, given by $W \equiv W(t) = \{(1/N)\sum_{i=1}^{N}(h_i - \langle h \rangle)^2\}^{1/2}$, where $h_i \equiv h_i(t)$ is the height of node i at time t, $\langle h \rangle$ is the mean value on the network, N is the system size, and $\{\cdot\}$ denotes an average over configurations. The EW process on networks is given by

$$\frac{\partial h_i}{\partial t} = \sum_{j=1}^{N} C_{ij}(h_j - h_i) + \eta_i, \tag{1}$$

where $C_{ij} = A_{ij} w_{ij}$ is a symmetric coupling strength, $\{A_{ij}\}$ is the adjacency matrix ($A_{ij} = 1$ if i and j are connected and zero otherwise), w_{ij} is the weight on the edge connecting i and j, and $\eta_i(t)$ is a Gaussian uncorrelated noise with zero mean and covariance $\{\eta_i \eta_j\} = 2D\delta_{ij}\delta(t - t')$. Here D is the diffusion coefficient and is taken in general as a constant. For nonweighted networks $w_{ij} = \nu =$const and thus Eq. (1) reduces to the unweighted EW equation on a graph given by $\partial h_i / \partial t = \nu \sum_{j=1}^{N} A_{ij}(h_j - h_i) + \eta_i$. Inspired by the results found for real networks where the weights are correlated with the topology, Korniss [7] studied synchronization for EW processes [see Eq. (1)] on SF networks where $w_{ij} = (k_i k_j)^{\beta}$ and k_i and k_j are the degrees of the nodes connected by a link. Using a mean-field approximation, he found that, subject to a fixed total edge cost, synchronization is optimal when $\beta = -1$, and at that point the performance is equivalent to that of the complete graph with the same edge cost. Pastore y Piontti *et al.* [1] used a discrete growth model with surface relaxation to the minimum (SRM) in SF networks, which mimics the fluctuation in the task-completion landscapes in certain distributed parallel schemes on computer networks, because it balances the load. They found that in SF networks with $\lambda < 3$ the saturation regime of $W \equiv W_s$ has a logarithmic divergence with N that cannot be explained with the unweighted EW equation in graphs, even though in Euclidean lattices the SRM model belongs to the same universality class as the EW equation [16].

In order to understand this discrepancy, in this paper we derive analytically the evolution equation for the SRM in random unweighted networks [1] and find that the dynamics introduces "weights" on the links. With our evolution equation, which contains non-linear terms in the height differences, we recover the logarithmic divergency of W_s with N found in [1] for SF networks with $\lambda < 3$. Let us first briefly recall the SRM discrete model [16], studied for SF networks by Pastore y Piontti *et al.* [1]. In this model, at each time step a node i is chosen with probability $1/N$. If we denote by v_i the nearest-neighbor nodes of i and $j \in v_i$, then (1) if $h_i \leq h_j \forall j \in v_i \Rightarrow h_i = h_i + 1$, else (2) if $h_j < h_n \forall n \neq j \in v_i \Rightarrow h_j = h_j + 1$. Next we derive the analytical evolution equation for the local height of the SRM model in random graphs. The procedure chosen here is based on a coarse-grained (CG) version of the discrete Langevin equations obtained from a Kramers-Moyal expansion of the master equation [17–19]. The discrete Langevin equation for the evolution of the height in any growth model is given by [18,19]

$$\frac{\partial h_i}{\partial t} = \frac{1}{\tau} G_i + \eta_i, \tag{2}$$

where G_i represents the deterministic growth rules that cause evolution of the node i, $\tau = N\delta t$ is the mean time to grow a layer of the interface, and η_i is a Gaussian noise with zero mean and covariance given by [18,19]

$$\{\eta_i(t)\eta_j(t')\} = \frac{1}{\tau} G_i \delta_{ij} \delta(t - t'). \tag{3}$$

We can write G_i more explicitly as

$$G_i = \omega_i + \sum_{j=1}^{N} A_{ij}\omega_j, \tag{4}$$

where ω_i is the growth contribution by deposition on node i and ω_j is the growth contribution to node i by relaxation from any of its j neighbors with

$$\omega_i = \prod_{j \in v_i} \Theta(h_j - h_i),$$

$$\omega_j = [1 - \Theta(h_i - h_j)] \prod_{n \in v_j} [1 - \Theta(h_i - h_n)].$$

Here, Θ is the Heaviside function given by $\Theta(x) = 1$ if $x \geq 0$ and zero otherwise, with $x = h_t - h_s \equiv \Delta h$. Without lost of generality, we take $\tau = 1$ and assume that the initial configuration of $\{h_i\}$ is random.

In the CG version $\Delta h \to 0$; thus after expanding an analytical representation of $\Theta(x)$ in Taylor series around $x = 0$ to second order in x, we obtain

$$\begin{aligned} G_i = {} & c_0^{k_i} + C_i + c_1 c_0^{k_i - 1} k_i \left(\sum_{j=1}^{N} \frac{A_{ij} h_j}{k_i} - h_i \right) + \frac{c_1}{(1 - c_0)} C_i \left(\sum_{j=1}^{N} \frac{C_{ij} h_j}{C_i} - h_i \right) + \frac{c_1}{(1 - c_0)} T_i \left(\sum_{j=1}^{N} \sum_{n=1, n \neq i}^{N} \frac{T_{ijn} h_n}{T_i} - h_i \right) \\ & - c_2 \sum_{j=1}^{N} A_{ij} \Omega(k_j - 1)(h_j - h_i)^2 - \left(c_2 + \frac{c_1^2}{2(1 - c_0)} \right) \sum_{j=1}^{N} A_{ij} \Omega(k_j - 1) \left(\sum_{n=1, n \neq i}^{N} A_{jn}(h_n - h_i)^2 \right) + c_0^{k_i - 1} \left(c_2 - \frac{c_1^2}{2c_0} \right) \sum_{j=1}^{N} A_{ij}(h_j - h_i)^2 \\ & + \frac{c_0^{k_i - 2} c_1^2}{2} \left(\sum_{j=1}^{N} A_{ij}(h_j - h_i) \right)^2 + \frac{c_1^2}{(1 - c_0)} \sum_{j=1}^{N} A_{ij} \Omega(k_j - 1)(h_j - h_i) \left(\sum_{n=1, n \neq i}^{N} A_{jn}(h_n - h_i) \right) + \frac{c_1^2}{2(1 - c_0)} \sum_{j=1}^{N} A_{ij} \Omega(k_j - 1) \\ & \times \left(\sum_{n=1, n \neq i}^{N} A_{jn}(h_n - h_i) \right)^2, \end{aligned} \tag{5}$$

where c_0, c_1, and c_2 are the first three coefficients of the expansion of $\Theta(x)$, $\Omega(k_j) = (1 - c_0)^{k_j}$ is the weight on the link ij introduced by the dynamic process, and

$$C_i = \sum_{j=1}^{N} C_{ij},$$

$$T_i = \sum_{j=1}^{N} \sum_{n=1, n \neq i}^{N} T_{ijn}, \tag{6}$$

with $C_{ij} = A_{ij}\Omega(k_j)$ and $T_{ijn} = A_{ij} A_{jn} \Omega(k_j)$.

In our equation the nonlinear terms in the difference of heights arise as a consequence of the lack of a geometrical direction and the heterogeneity of the underlying network. This result is very different from the one found in Euclidean lattices, where for the SRM model the nonlinear terms disappear due to the symmetry of the process and the homogeneity of the lattice.

For the noise correlation [see Eq. (3)], up to zero order in Δh [18,19] we obtain $\{\eta_i(t)\eta_j(t')\} = 2D(k_i)\delta_{ij}\delta(t - t')$ with

$$D(k_i) = \frac{1}{2}(c_0^{k_i} + C_i). \tag{7}$$

Notice that all the coefficients of the equation depend on the connectivity of node i, i.e., on the network topology of the underlying network. This dependence on the topology can be thought of as a weight on the links of the unweighted under-

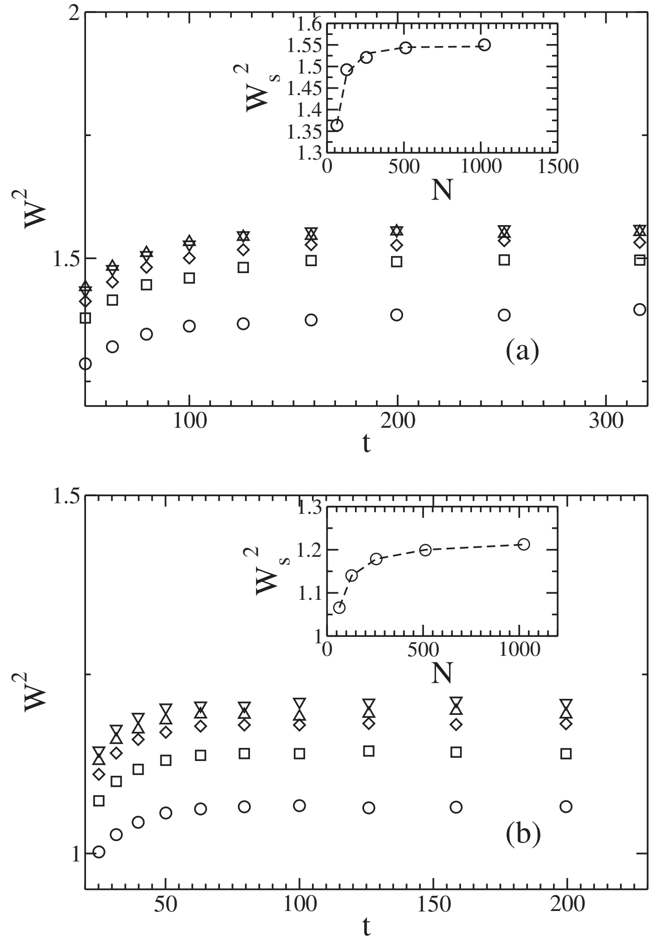

FIG. 1. W^2 as a function of t from the integration of the evolution equation using the linear terms for $N=256$ (○), 384 (□), 512 (◇), 768 (△), and 1024 (▽). $\lambda=$ (a) 3.5 and (b) 2.5. In the inset figure we plot W_s^2 vs N in symbols. The dashed lines represent the fitting with Eq. (12), obtained by considering the finite-size effects introduced by the MR construction. For all the integrations we used $U=0.5$ and typically 10 000 realizations of networks.

lying network that appears only due to the dynamics on the heterogeneous network.

Interestingly, the linear terms are different from the EW process as shown below. Keeping only the linear terms in Eq. (5), we numerically integrate our evolution equation in a SF network using the Euler method with the representation of the Heaviside function given by $\Theta(x)=\{1+\tanh[U(x+z)]\}/2$, where U is the width and $z=1/2$ [19]. With this representation $c_0=[1+\tanh(U/2)]/2$, $c_1=[1-\tanh^2(U/2)]U/2$, and $c_2=[-\tanh(U/2)+\tanh^3(U/2)]U^2/2$. We build the network using the Molloy-Reed (MR) algorithm [20]. In Fig. 1, we plot W^2 as a function of t, obtained from the integration of Eq. (2) using only the linear terms of Eq. (5) with $D(k_i)$ given by Eq. (7) for $\lambda=3.5$ and 2.5 and different values of N with $k_{\min}=2$ in order to ensure that the network is fully connected. For the time step integration we chose $\Delta t \ll 1/k_{\max}$ according to Ref. [21]. In contrast to the results obtained for the EW process [1], W_s increases with N until it reaches a constant value. As shown below, this dependence of W_s on N is due to finite-size effects due to the MR construction.

Now we apply a mean-field approximation to the linear terms of Eq. (5). In this approximation we consider $1 \ll k_{\min} \ll k_{\max}$ and disregard the fluctuations. Then $\sum_{j=1}^{N} A_{ij} h_j / k_i \approx \langle h \rangle$, $\sum_{j=1}^{N} C_{ij} h_j / C_i \approx \langle h \rangle$, and $\sum_{j=1}^{N} \sum_{n=1, n\neq i}^{N} T_{ijn} h_n / T_i \approx \langle h \rangle$. Multiplying and dividing Eq. (6) by k_i, we can approximate C_i by $C_i(k_i) \approx k_i \int_{k_{\min}}^{k_{\max}} P(k|k_i)\Omega(k)dk$ [7], where $P((k|k_i))$ is the probability that a node with degree k_i is connected to another with degree k. For uncorrelated networks, $P((k|k_i))=kP(k)/\langle k \rangle$ [8] does not depend on k_i; then $C_i(k_i) \approx I_1 k_i / \langle k \rangle$ with $I_1 = \int_{k_{\min}}^{k_{\max}} P(k) k \Omega(k) dk$. Making the same assumption for T_i, we obtain $T_i(k_i) \approx I_2 k_i / \langle k \rangle$ with $I_2 = \int_{k_{\min}}^{k_{\max}} P(k) k (k-1) \Omega(k) dk$. Then the linearized evolution equation for the heights can be written as

$$\frac{\partial h_i}{\partial t} = F_i(k_i) + \nu_i(k_i)(\langle h \rangle - h_i) + \eta_i, \tag{8}$$

where $F_i(k_i)=c_0^{k_i}+k_i I_1/\langle k \rangle$ represents a local driving force, $\nu_i(k_i)=(c_1 c_0^{k_i-1}+b)k_i$ is a local superficial tensionlike coefficient with $b=c_1(I_1+I_2)/\langle k \rangle$, and η_i is a Gaussian noise with covariance $D(k_i)=F_i(k_i)/2$. This approximation shows the full topology of the network through $P(k)$.

Taking the average over the network in Eq. (8), $\partial \langle h \rangle / \partial t = (1/N)\sum_{i=1}^{N} F_i = F$; then $\langle h \rangle = Ft$ is linear with t. The solution of Eq. (8) [17] is given by

$$h_i(t) = \int_0^t e^{-\nu_i(t-s)}[F_i + \nu_i \langle h(s) \rangle + \eta_i(s)]ds$$
$$= \left(\frac{F_i - F}{\nu_i}\right) - \left(\frac{F_i - F}{\nu_i}\right) e^{-\nu_i t} + \langle h \rangle + \int_0^t e^{-\nu_i(t-s)} \eta_i(s) ds. \tag{9}$$

Using Eq. (9), the two-point correlation function for $t > \max\{1/\nu_i\} \sim 1/k_{\min}$, is

$$\{[h_i(t_1) - \langle h \rangle][h_j(t_2) - \langle h \rangle]\} = \left(\frac{F_i - F}{\nu_i}\right)\left(\frac{F_j - F}{\nu_j}\right) + \int_0^{t_2}\int_0^{t_1} e^{-\nu_i(t_1-s_1)} e^{-\nu_j(t_2-s_2)} \times \{\eta_i(s_1)\eta_j(s_2)\} ds_1 ds_2.$$

Then W_s can be written as

$$W_s^2 = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{F_i - F}{\nu_i}\right)^2 + \frac{1}{N}\sum_{i=1}^{N}\frac{2D(k_i)}{2\nu_i}. \tag{10}$$

For SF networks it can be shown that $I_1, I_2 \sim \text{const} + k_{\max}\exp(-k_{\max} \times \text{const})$, where $k_{\max} \sim N^{1/(\lambda-1)}$ for MR networks; thus finite-size effects due to the cutoff on these quantities can be disregarded. Replacing in the last equation $D(k_i)$ by $F_i(k_i)/2$, we obtain

$$W_s^2 \sim \left(1 - 2\langle k \rangle \left\langle \frac{1}{k} \right\rangle + \langle k \rangle^2 \left\langle \frac{1}{k^2} \right\rangle\right) + \text{const}. \tag{11}$$

Notice that, if $F_i=0$, $D=\text{const}$, and $\nu_i \propto k_i$, we recover the EW equation found in [7]. Using the corrections due to finite-size effects introduced by k_{max} [1] in Eq. (11),

$$W_s^2 \sim W_s^2(\infty)\left(1 + q_1 \frac{1}{N^{\lambda-2/\lambda-1}} + q_2 \frac{1}{N}\right), \qquad (12)$$

where $W_s^2(\infty) = W_s^2(N \to \infty)$ and q_1 and q_2 are constants. In the inset of Figs. 1(a) and 1(b) we plot W_s^2 as function on N and the fitting obtained from Eq. (12). The agreement with the scaling form, Eq. (12), is excellent. Thus, the linear approximation can only explain the finite-size effects due to the MR construction but fails to predict the logarithmic divergency of W_s with N for $\lambda < 3$ found in Ref. [1]. Next we show that the nonlinear terms are responsible for this behavior. We integrate our evolution equation for SF networks with the linear terms and only the first nonlinear term [see Eq. (5)] due to the numerical instability produced when we try to incorporate all of them. Even with only one nonlinear term, we recover the logarithmic divergency of W_s with N for $\lambda < 3$. The results of the integration are shown in Fig. 2, where we plot W as a function of t for (a) $\lambda = 3.5$ and (b) $\lambda = 2.5$ and different values of N. In the inset figures we plot W_s as a function of N. We can see that, for $\lambda = 3.5$, W_s increases but asymptotically goes to a constant and all the N dependence is due to finite-size effects. However, for $\lambda = 2.5$ we found a logarithmic divergency of W_s with N [1], as shown in the inset of Fig. 2(b), where we plot W_s as a function of N on a log-linear scale. The fit of W_s with a logarithmic function for $\lambda = 2.5$ shows the agreement between our results and those obtained for the SRM model in SF networks for $\lambda < 3$. Discrepancies between behaviors in regular Euclidean lattices and Euclidean lattices after addition of random links were found before in [22].

In summary, we derived analytically the evolution equation for the SRM model and found, surprisingly, that even when the underlying network is unweighted the dynamics introduces weights on the links that depend on the topology. We also found that the linear terms can explain only finite-size effects due to the MR construction. The linear mean-field approximation shows clearly the effects of the topology on the dynamics and the corrections due to finite-size effects. When nonlinear terms on SF networks are considered, new numerical integration algorithms are needed in order to avoid numerical instabilities. This is still an open problem to be solved in the future. With all the linear terms and one nonlinear term, we recovered the logarithmic divergency of W_s with N of the SRM model for $\lambda < 3$. Our analytic procedure can be also applied to any other growth model.

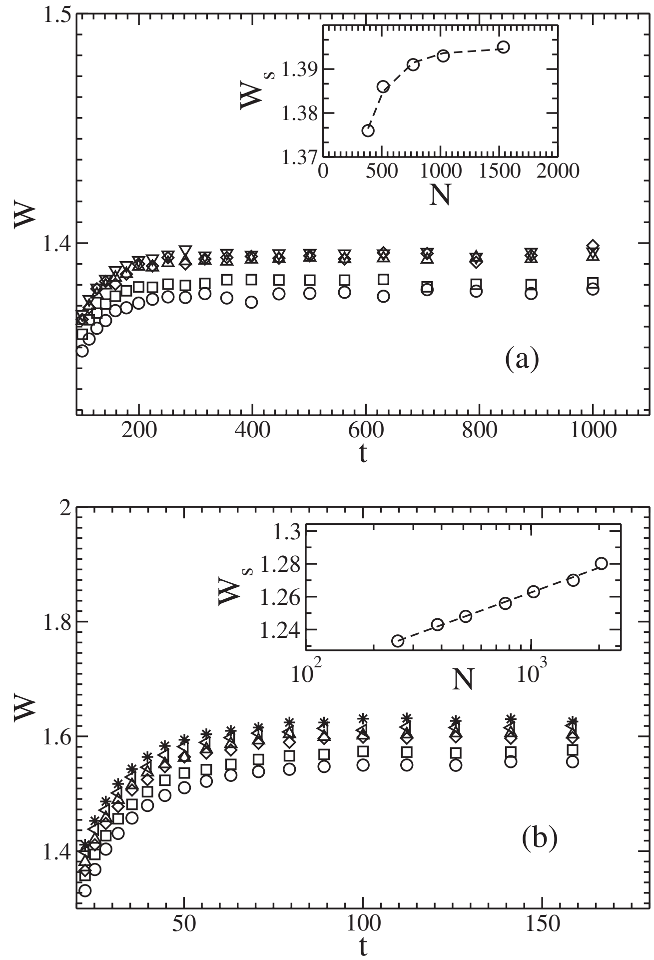

FIG. 2. W as a function of t from the integration of the evolution equation using the linear terms and the first nonlinear term for $N = 384$ (○), 512 (□), 768 (◇), 1024 (△), and 1536 (▽). $\lambda =$ (a) 3.5 and (b) 2.5. In the inset figure we plot W_s vs N in symbols. The dashed lines represent the fitting with Eq. (12) in (a) and $W_s \sim \ln N$ in (b).

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